summary: exercises about stochastic calculus

22. Let $M = \{M(t), t \geq 0\}$, with

$$M(t) = \cancel{\frac{1}{\sqrt{1-t}}} \exp\left(-\frac{W^2(t)}{2(1-t)}\right)$$

a) Derive a stochastic differential equation such that M is a solution.

b) Show that M is a martingale.

c) Compute $\mathbb{E}[M(t)]$.

$dx(\tau) = \mu(x(\tau))\,d\tau + \sigma(x(\tau))\,dw(\tau)$

a) $dM(t) = ?$

$dx(\tau) = dw(\tau) \Rightarrow \mu = 0;\ \sigma = 1$

$f(\tau, x) = \exp\left(-\frac{x^2}{2(1-\tau)}\right)$

Itô's formula:

$$df(t, x(\tau)) = \Big[\underbrace{\frac{\partial f}{\partial \tau}}_{f_\tau} + \mu(x(\tau))\underbrace{\frac{\partial f}{\partial x}}_{f_x} + \frac{1}{2}\sigma^2(x(\tau))\underbrace{\frac{\partial^2 f}{\partial x^2}}_{f_{xx}}\Big]\,d\tau$$

$$+\ \sigma(x(\tau))\frac{\partial f}{\partial x}\,dw(\tau)$$

$$\frac{\partial f}{\partial \tau} = -\frac{x^2}{2}\left(\frac{1}{1-\tau}\right)' \exp\left(-\frac{x^2}{2(1-\tau)}\right)$$

$$= -\frac{x^2}{2}\frac{1}{(1-\tau)^2}\underbrace{\boxed{\exp\left(-\frac{x^2}{2(1-\tau)}\right)}}_{M(\tau)}$$

$x \to w(\tau)$

$$\frac{\partial f}{\partial x} = -\frac{x}{(1-\tau)}\exp\left(-\frac{x^2}{2(1-\tau)}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{1-\tau}\underbrace{\boxed{\exp\left(-\frac{x^2}{2(1-\tau)}\right)}}_{M(t)} + \left(\frac{x}{1-\tau}\right)^2\underbrace{\boxed{\exp\left(-\frac{x^2}{2(1-\tau)}\right)}}_{M(\tau)}$$

Therefore:

$$dM(\tau) = \left[-\frac{w^2(\tau)}{2}\left(\frac{1}{1-\tau}\right)^2 M(\tau) + \frac{1}{2}\left(\left(\frac{w^2(t)}{1-\tau}\right)^2 - \frac{1}{1-\tau}\right) M(\tau)\right] d\tau - \frac{w(\tau)}{(1-\tau)} M(t)\, dw(\tau)$$

ie:

$$dM(t) = M(\tau)\left(\frac{1}{2}\left(\frac{w^2(t)}{1-\tau}\right)^2 - \frac{1}{2}\frac{1}{1-\tau} - \frac{w^2(\tau)}{2}\left(\frac{1}{1-\tau}\right)^2\right) d\tau$$
$$- \frac{w(\tau) M(\tau)}{1-\tau}\, dw(\tau)$$

b) $\{M(\tau), \tau \geq 0\}$ is a martingale?
No, because dM has a non-zero drift term.

☒

Consider the transformation:

$M(\tau) = \exp(-w^2(\tau))$; $dM(\tau) = ?$

we have two ways to solve this exercise:

i) $dx(\tau) = dw(\tau) \Rightarrow \mu = 0;\ \sigma = 1 \quad (x(\tau) = w(t))$

$f(x) = \exp(-x^2)$

$f'(x) = -2x \exp(-x^2)$

$f''(x) = -2 \underbrace{\exp(-x^2)}_{M(\tau)} + 4x^2 \underbrace{\exp(-x^2)}_{M(\tau)}$

$$dM(\tau) = \left[0 + 0 + \frac{1}{2}\left(-2 + 4w^2(\tau)\right) M(t)\right] d\tau$$
$$- 2w(\tau) M(t)\, dw(\tau)$$

ie: $$\boxed{dM(\tau) = \left(-1 + 2w^2(t)\right) M(\tau)\, d\tau - 2w(\tau) M(\tau)\, dw(\tau)}$$

ii) $x(\tau) = w^2(\tau) \Rightarrow \mu = ?\quad \sigma = ?\quad (dx(\tau) = dw^2(t))$

$dw^2(\tau) = w(\tau)\, dw(\tau) + w(\tau)\, dw(\tau) + dw(t)\, dw(\tau)$

$$\left[\, d(xy) = dx\, y + x\, dy + dx\, dy \,\right] \qquad (dW(t))^2 = dt$$

$$= dt + 2W(t)\, dW(t)$$

$$\Rightarrow \mu = 1 \;;\; \sigma = 2W(t)$$

$$g(y) = \exp(-y)$$

$$g'(y) = -\exp(-y) \to -M(t)$$

$$g''(y) = \exp(-y) \to M(t)$$

$$dM(t) = \left[-M(t) + \frac{1}{2}(2W(t))^2 M(t) \right] dt - 2W(t)M(t)\, dW(t)$$

$$= (-1 + 2W^2(t))\, M(t)\, dt - 2W(t)M(t)\, dW(t)$$

13. Let $X = \{X(t) = W^3(t) - 3tW(t), t \geq 0\}$. Prove, using the definition of martingale, that X is a martingale (note that if X is a r.v. with symmetric density distribution, then $\mathbb{E}[X^3] = 0$).

$W(t) \sim N(0, t)$

$E[W^3(t)] = 0$

• By definition: for $s < t$:

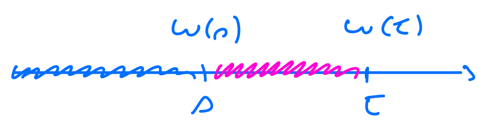
$$E[X(t) \mid \mathcal{F}_s^W] = X(s)$$

where $\mathcal{F}_s^W$ is the σ-algebra generated by the BM until time s.

$$E[W^3(t) - 3tW(t) \mid \mathcal{F}_s^W] \overset{?}{=} W^3(s) - 3sW(s)$$

$$\left(E[W^3(t) - 3tW(t)] = E[W^3(t)] - 3tE[W(t)] = \right.$$

$$\left. = 0 - 3t \times 0 = 0 \right)$$



$$= E\left[(\underbrace{W(t) - W(s)} + W(s))^3 - 3t(W(t) - W(s) + W(s)) \mid \mathcal{F}_s^W \right]$$

$$= E\left[\underbrace{(W(t) - W(s))^3}_{a} + \underbrace{3(W(t) - W(s))^2 W(s)}_{b} \right.$$

$$\left. + \underbrace{3(W(t) - W(s))W^2(s)}_{c} + \underbrace{W^3(s)}_{d} \mid \mathcal{F}_s^W \right]$$

$$-3\tau\, E\Big[\big(\underbrace{w(\tau)-w(s)}_{e}\big) + \underbrace{w(s)}_{f} \,\Big|\, \mathcal{F}_s^w\Big] = \circledast$$

a) independence of increments: $E\left[\left(w(\tau)-w(s)\right)^3\right]$

b) independence of increments + $w(s)$ is $\mathcal{F}_s$-measurable:

$$3\,w(s)\,E\left[\left(w(\tau)-w(s)\right)^2\right]$$

c) '' '' '' ''

$$3\,w^2(s)\,E\left[w(\tau)-w(s)\right]$$

d) $w(s)$ is $\mathcal{F}_s$-measurable : $w^3(s)$

e) independence of increments: $-3\tau\, E\left[w(\tau)-w(s)\right]$

f) $-3\tau\, w(s)$

$$\circledast = E\Big[\big(\underbrace{\cancel{w(t)-w(s)}}_{=0}\big)^3\Big] + 3w(s)\,\overbrace{E\left[\left(w(t)-w(s)\right)^2\right]}^{(t-s)}$$

$$+\,3w^2(s)\,E\big[\underbrace{\cancel{w(\tau)-w(s)}}_{=0}\big] + w^3(s)$$

$$-\,3t\,E\big[\underbrace{\cancel{w(\tau)-w(s)}}_{=0}\big] - 3\tau\, w(s)$$

$w(\tau) \sim \mathcal{N}(0,\tau)$

$w(\tau)-w(s) \sim \mathcal{N}(0,\tau-s)$

$$= 3w(s)\,(\cancel{t}-s) + w^3(s) - 3\cancel{\tau w(s)}$$

$$= w^3(s) - 3s\,w(s)$$

which means that it is a martingale!

- By deriving the drift term of

$$d\left(w^3(t)-3\tau w(\tau)\right) = d\,w^3(\tau) - 3\,d\left(\tau w(\tau)\right)$$

$dx(\tau) = dw(\tau)$ $\quad \mu = 0\ ;\ \sigma = 1$

$f(t,x) = x^3 - 3\tau x$

$$\frac{\partial f}{\partial t} = -3x \qquad \frac{\partial f}{\partial x} = 3x^2 - 3t \qquad \frac{\partial^2 f}{\partial x^2} = 6x$$

$$d\left[W^3(t) - 3tW(t)\right] = \left[-3W(t) + \frac{1}{2}6W(t)\right]dt + \left(3W^2(t) - 3t\right)dW(t)$$

$$= 3\left(W^2(t) - t\right)dW(t)$$

which, again, proves that this process is a martingale!

20. Consider the processes $\{X(t) : t \geq 0\}$ and $\{B(t) : t \geq 0\}$, where

$$X(t) = W(t) - tW(1)$$
$$B(t) = (t+1)X\left(\tfrac{t}{t+1}\right)$$

Show that $\{B(t), t \geq 0\}$ is also a Brownian motion.

- $B(0) = 0$
- $B(t) \sim \mathcal{N}(0, t)$
- independence of increments
- $B(t) - B(s) \sim \mathcal{N}(0, t-s)$
- continuous sample-paths

- $B(0) = X(0) = W(0) - 0W(1) = 0$ ✓
- $\{B(t), t \geq 0\}$ has continuous sample-paths because it is a product of a continuous function $(t+1)$ with the difference of two continuous functions ✓
- $W(t) \sim \mathcal{N}(0, t)$

$A \sim \mathcal{N}(\mu_A, \sigma_A^2) \qquad B \sim \mathcal{N}(\mu_B, \sigma_B^2)$

$A + B \sim \mathcal{N}(\quad)$

$A - B \sim \mathcal{N}(\quad)$

$X(t) = W(t) - tW(1)$
$B(t) = (t+1)X(\frac{t}{t+1})$

$W(t) \sim N(0,t)$

$W(1) \sim N(0,1)$

$\Rightarrow -tW(1) \sim N(0,t^2)$

Thus: $W(t) - tW(1) \sim N(0,\sigma^2)$

$\sigma^2 = var(W(t) - tW(1))$

$= var(W(t)) + var(tW(1))$

$- 2\,cov(W(t), tW(1))$

$(var(aX) = a^2 var(X))$

$var(X+Y) = var(X) +$

$+ var(Y) + 2cov(X,Y)$

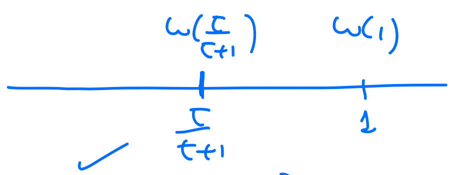
$var(X-Y) = var(X)$

$+ Var(Y) - 2cov(X,Y)$

Thus $B(t) = (t+1)\, N(0, ?) = N(0, (t+1)^2 ?)$
and hence it is also normally distributed

$var(B(t)) = (t+1)^2 var\left(X\left(\frac{t}{t+1}\right)\right)$

$X\left(\frac{t}{t+1}\right) = W\left(\frac{t}{t+1}\right) - \left(\frac{t}{t+1}\right) W(1)$



$var\left(X\left(\frac{t}{t+1}\right)\right) = var\left(W\left(\frac{t}{t+1}\right)\right) + var\left(\frac{t}{t+1} W(1)\right)$

$- 2\,cov\left(W\left(\frac{t}{t+1}\right), \frac{t}{t+1} W(1)\right)$

- $var\left(W\left(\frac{t}{t+1}\right)\right) = \frac{t}{t+1}$
- $var\left(\frac{t}{t+1} W(1)\right) = \left(\frac{t}{t+1}\right)^2 var(W(1)) = \left(\frac{t}{t+1}\right)^2$
- $cov\left(W\left(\frac{t}{t+1}\right), \frac{t}{t+1} W(1)\right) = \frac{t}{t+1}\, cov\left(W\left(\frac{t}{t+1}\right), W(1)\right)$

$cov(W(s), W(u)) = s$ $\quad = \frac{t}{\,}\ \frac{t}{\,} = \left(\frac{t}{\,}\right)^2$

$\tau+1 \quad \tau+1 \quad (\tau+1)$

$s < u$

So: $\operatorname{var}\left(X\left(\frac{\tau}{\tau+1}\right)\right) = \frac{\tau}{\tau+1} + \left(\frac{\tau}{\tau+1}\right)^2 - 2\left(\frac{\tau}{\tau+1}\right)^2$

$$= \frac{\tau}{\tau+1}\left[1 - \frac{\tau}{\tau+1}\right] = \frac{\tau}{\tau+1}\,\frac{\cancel{\tau}+1-\cancel{\tau}}{\tau+1}$$

$$= \frac{\tau}{(\tau+1)^2}$$

$$\operatorname{var}(B(t)) = (t+1)^2 \operatorname{var}\left(X\left(\frac{t}{t+1}\right)\right) = (t+1)^2\,\frac{t}{(t+1)^2} = t \;//$$

$$\Rightarrow B(t) \sim N(0, t)$$

In order to prove that you have independence increments, as we have already proved that it is normally distributed, we just need to prove that

$$\operatorname{Cov}(B(s), B(t) - B(s)) = 0$$